Closing Today: 3.5(1)(2)
Closing *Tues*: 3.6-9
Closing *Thurs*: 3.9

Entry Task: (from an old final)

Find the derivative of

$$y = x^{\sin^{-1}(2x)}$$

or

$$y = e^{\sin^{-1}(2x)\ln(x)}$$

$$\ln(y) = \sin^{-1}(2x)\ln(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin^{-1}(2x)\frac{1}{x} + \frac{1}{\sqrt{1-(2x)^2}}\cdot 2\ln(x)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin^{-1}(2x)}{x} + \frac{2\ln(x)}{\sqrt{1-4x^2}}\right)$$

$$\boxed{= x^{\sin^{-1}(2x)}\left(\frac{\sin^{-1}(2x)}{x} + \frac{2\ln(x)}{\sqrt{1-4x^2}}\right)}$$

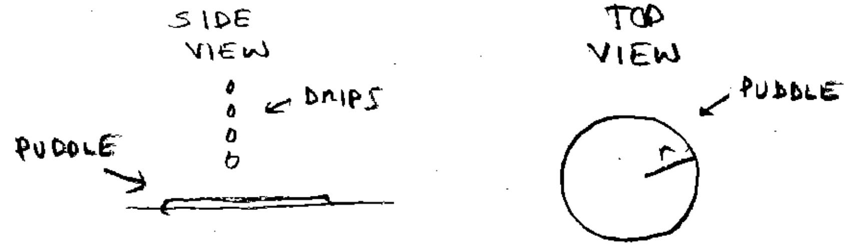
3.9: Related Rates

Motivation: In an application, it is common that the rate of change of one quantity is known. A ***related rates*** question asks for other rates in the application.

The key to these problems is to find general relationships between the quantities, then differentiating to find the relationship between the rates.

NEW IDEA

A Simple Example: Water is dripping on the ground forming a circular puddle. The area of the puddle is growing at a constant 20 in^2/min. At what rate is the radius increasing when the radius is 5 in?



$r = r(t) =$ radius in

$A = A(t) =$ area in^2

$A = \pi r^2$ IS ALWAYS TRUE

$$\frac{d}{dt}\left[A = \pi r^2\right] \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

- WE KNOW $\frac{dA}{dt} = 20 \frac{\text{in}^2}{\text{min}}$
- WE WANT $\frac{dr}{dt} = ???$ when $r = 5$ in

$$\Rightarrow \quad 20 = 2\pi(5)\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{2}{\pi} \approx 0.6366 \frac{\text{in}}{\text{min}}$$

Recipe for Related Rates:

1. Draw a good picture.
   Label ***everything***.
2. Identify what you **know**?
   Identify what you **want**?
3. Write equations relating the labels.
4. *Implicitly differentiate* with respect to time t. (Treat all changing quantities as functions of t)
5. Substitute in your values and solve.
   Do **<u>NOT</u>** substitute values in until the last step.

Tools to use (for *step 3*):

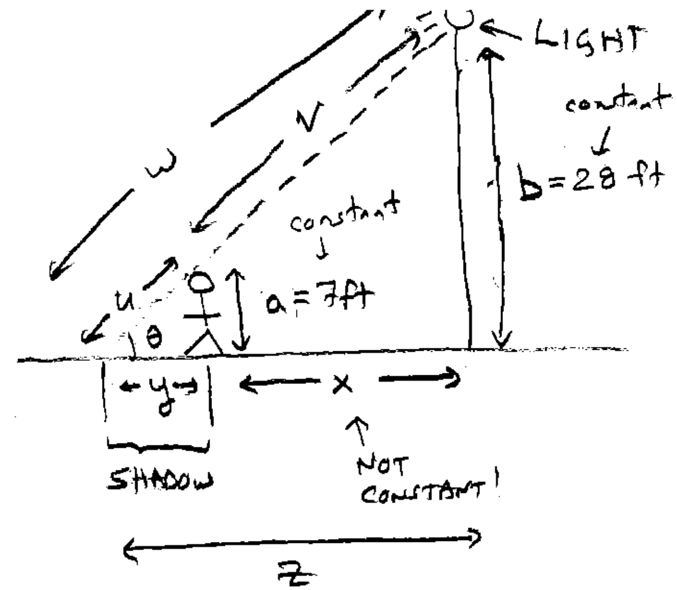
- Area of circles, squares, triangles.
- Volume of a cylinder, cone, sphere.
- Surface area of a sphere.
- Pythagorean Theorem.
- Similar Triangles.
- Trig. Definitions (Soh, Cah Toa).
  - (Note: If you don't have a right triangle, make right triangles)
- Trig. Identity (Cosine/Sine Laws).

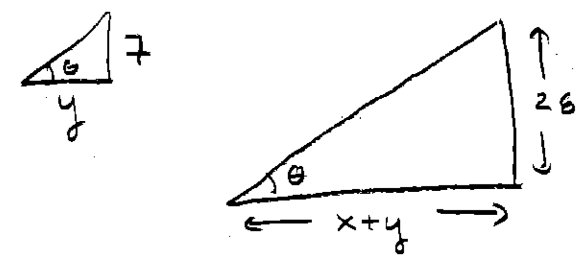
Note: Once you have the general idea, these problems become routine and they are <u>all the same</u>. So don't think of each problem as different, recognize they are the same idea. Now we will do a bunch of examples.

Example: (Like HW 3.9/1)

A man 7 ft tall is 20 ft from a 28-ft lamppost and is walking toward it at a rate of 4 ft/sec.

- How fast is his shadow shrinking at that moment?
- How fast is the tip of the shadow moving?





KNOW : $\frac{dx}{dt} = -4$ ft/sec when $x = 20$ ft

WANT : $\frac{dy}{dt}$ = "HOW FAST SHADOW SHRINKING"

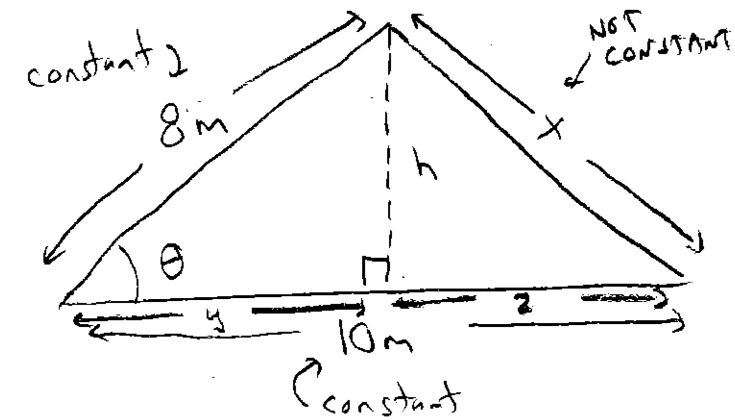
$\frac{dz}{dt}$ = "HOW FAST TIP OF THE SHADOW MOVING"

SIMILAR TRIANGLES : $\frac{y}{7} = \frac{x+y}{28} \Rightarrow 4y = x + y \Rightarrow 3y = x$

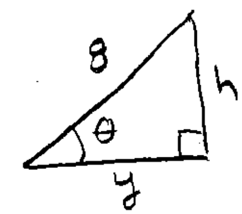
$\Rightarrow 3\frac{dy}{dt} = \underbrace{\frac{dx}{dt}}_{-4} \Rightarrow \boxed{\frac{dy}{dt} = -\frac{4}{3}\ \frac{ft}{sec}}$

$z = x + y \Rightarrow \frac{dz}{dt} = \underbrace{\frac{dx}{dt}}_{-4} + \underbrace{\frac{dy}{dt}}_{-4/3} = -4 - 4/3 = \boxed{-\frac{16}{3}\ \text{ft/sec}}$

Example: (Like HW 3.6-9/11)
Two sides of a triangle are 8 m and 10 m in length and the angle between them is increasing at a rate of 0.06 rad/s.
Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$ radians.



Let $A = A(t) =$ Area of the triangle
$\theta = \theta(t) =$ angle
$h = h(t) =$ height



CAN YOU THINK OF A RELATIONSHIP BETWEEN θ AND h???

$\sin\theta = \frac{h}{8}$ ALWAYS

$\Rightarrow h = 8\sin\theta$

KNOW: $\frac{d\theta}{dt} = 0.06 \ \frac{\text{RAD}}{\text{SEC}}$

WANT: $\frac{dA}{dt} = \ ??? \ \frac{m^2}{\text{SEC}}$ when $\theta = \pi/3$

ALWAYS!

$$A = \frac{1}{2}bh = \frac{1}{2}(10)h \Rightarrow A = 5h$$

$$\Rightarrow A = 5 \cdot 8\sin(\theta)$$

$$A = 40\sin(\theta)$$

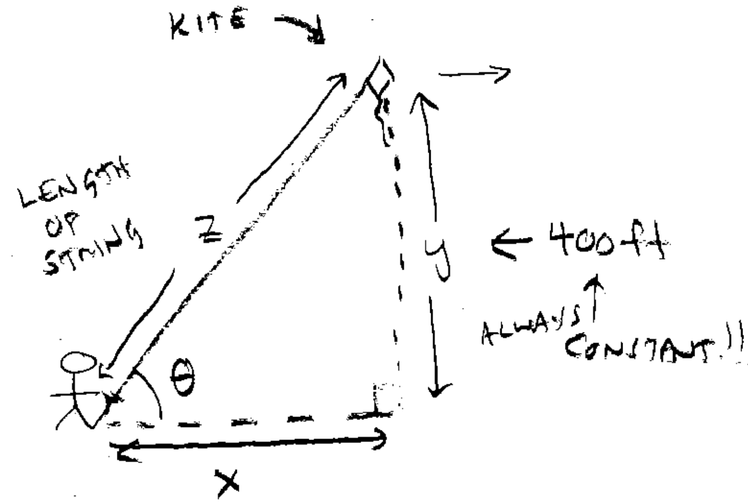
$$\frac{dA}{dt} = 40(\underset{\uparrow \atop \pi/3}{\cos\theta})\underset{\uparrow \atop 0.06}{\frac{d\theta}{dt}}$$

$$\Rightarrow \frac{dA}{dt} = 40 \cdot \frac{1}{2} \cdot 0.06 = \boxed{1.2 \ \frac{m^2}{\text{sec}}}$$

Example: (Like HW 3.9/3)

A kite in the air at an altitude of 400 ft is being blown horizontally at the rate of 10 ft/sec away from the person holding the kite string at ground level.

At what rate is the string being let out when 500 ft of string is already out?



$x = x(t)$ = horizontal distance

$z = z(t)$ = amount of string let out

KNOW: $\frac{dx}{dt} = 10 \frac{ft}{sec}$

WANT: $\frac{dz}{dt} = ??? \frac{ft}{sec}$ WHEN $z = 500$ ft

$$x^2 + 400^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$$

$$\Rightarrow \underset{???}{x}\,\underset{10}{\frac{dx}{dt}} = \underset{500}{z}\,\frac{dz}{dt} \Rightarrow 300 \cdot 10 = 500 \cdot \frac{dz}{dt}$$

$$6 = \frac{dz}{dt}$$

$$\boxed{6 \text{ ft/sec}}$$

SOLVE FOR x NOW

$x^2 + 400^2 = 500^2$

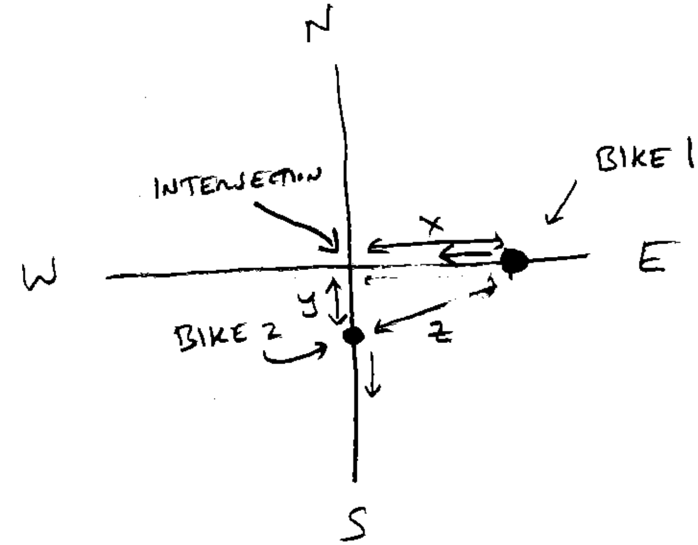
$\Rightarrow x = 300$

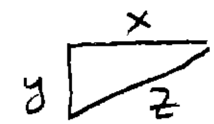
Example: (Like HW 3.9/2)

One bike is 4 miles east of an intersection, travelling toward the intersection at the rate of 9 mph.

At the same time, a 2nd bike is 3 miles south of the intersection and is travelling away from the intersection at a rate of 10 mph.

- At what rate is the distance between them changing?
- Is this distance increasing or decreasing?





KNOW: $\frac{dx}{dt} = -9$ WHEN $x = 4$

$\frac{dy}{dt} = 10$ WHEN $y = 3$

WANT: $\frac{dz}{dt} = ???$ WHEN $x=4, y=3$

$$z^2 = x^2 + y^2$$

$$\Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\underbrace{z}_{???}\frac{dz}{dt} = \underbrace{x}_{4}\underbrace{\frac{dx}{dt}}_{(-9)} + \underbrace{y}_{3}\underbrace{\frac{dy}{dt}}_{(10)}$$

$z^2 = x^2 + y^2$

$z^2 = 4^2 + 3^2 \Rightarrow z = 5$

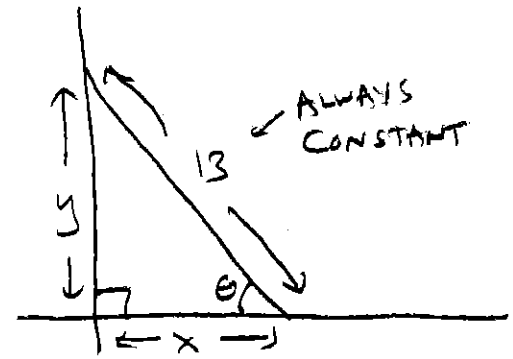
$$5\frac{dz}{dt} = -36 + 30 = -6$$

$$\boxed{\frac{dz}{dt} = -\frac{6}{5} \text{ mph}}$$

DECREASING!!

Example: (Like 3.6-9/13, 3.9/9)

A 13-foot ladder is leaning against a wall and its base is slipping away from the wall at a rate of 3 ft/sec when it is 5 ft from the wall.



How fast is the top of the ladder dropping at that moment?

KNOW : $\frac{dx}{dt} = 3$ WHEN $x = 5$

WANT : $\frac{dy}{dt} = ?$ WHEN $x = 5$

$x^2 + y^2 = 13^2 = 169$

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow (5)(3) + (12)\frac{dy}{dt} = 0$

$\uparrow \quad \uparrow \quad \uparrow$

$5 \quad 3 \quad$???

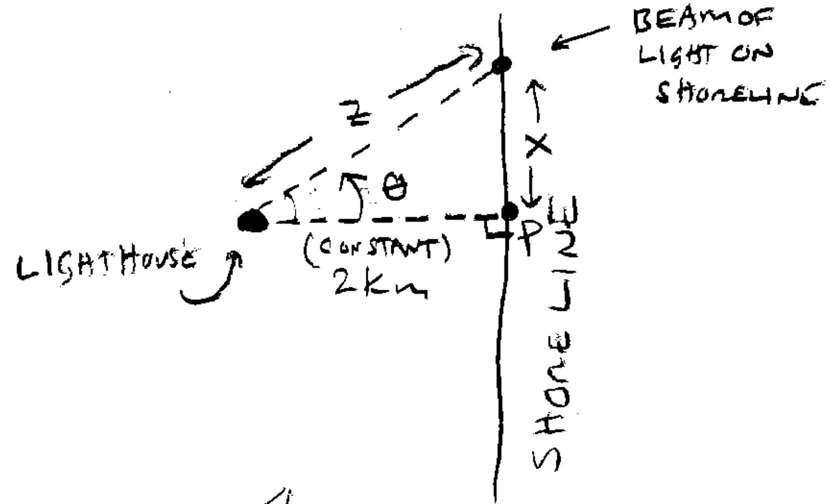
$5^2 + y^2 = 13^2 \Rightarrow y^2 = 169 - 25 = 144$

$y = 12$

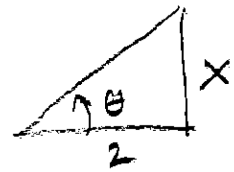
$$\boxed{\Rightarrow \frac{dy}{dt} = -\frac{15}{12} = -\frac{5}{4} = -1.25 \text{ ft/sec}}$$

Example: (Like 3.9/6)

A lighthouse is located on a small island 2 km away from the nearest point *P* on a straight shoreline and its light makes three revolutions per minute.



How fast is the beam of light moving along the shoreline when it is 1 km from *P*?



KNOW: $\dfrac{d\theta}{dt} = \dfrac{3 \text{ REV}}{\text{MIN}} = \dfrac{6\pi \text{ RAD}}{\text{MIN}}$

WANT: $\dfrac{dx}{dt} = ???$ WHEN $x = 1$ km

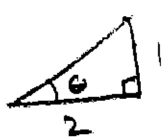
$\tan\theta = \dfrac{x}{2} \Rightarrow x = 2\tan\theta \Longrightarrow \dfrac{dx}{dt} = 2\underbrace{\sec^2\theta}_{???\text{ WHEN } x=1} \underbrace{\dfrac{d\theta}{dt}}_{6\pi}$

$= 2\left(\dfrac{\sqrt{5}}{2}\right)^2 6\pi = 2\,\dfrac{5}{4}\cdot 6\pi$

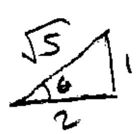
$= \boxed{15\pi \approx 47.124 \text{ km/min}}$

$x = 1 \Rightarrow 1 = 2\tan\theta$

$\Rightarrow \frac{1}{2} = \tan\theta$



$\Rightarrow$



$\Rightarrow \cos\theta = \dfrac{2}{\sqrt{5}}$

$\Rightarrow \sec\theta = \dfrac{\sqrt{5}}{2}$